# Key to Tutorial 4
# Boolean Algebra

## Exercise 1

Simplify the expressions below by using the algebraic method.

- $S1 = (a + b).(\overline{a} + \overline{b})$
  $S1 = a.\overline{a} + a.\overline{b} + b.\overline{a} + b.\overline{b}$
  $S1 = 0 + a.\overline{b} + b.\overline{a} + 0$
  $S1 = a.\overline{b} + \overline{a}.b$
  $S1 = a \oplus b$

- $S2 = \overline{a.b + \overline{a}.\overline{b} + \overline{a}.b}$ → *De Morgan's Theorem*
  $S2 = \overline{a.b} \,.\, \overline{\overline{a}.\overline{b}} \,.\, \overline{\overline{a}.b}$ → *De Morgan's Theorem*
  $S2 = (\overline{a} + \overline{b}).(a + b).(a + \overline{b})$
  $S2 = (\overline{a}.a + \overline{a}.b + a.\overline{b} + b.\overline{b}).(a + \overline{b})$
  $S2 = (0 + \overline{a}.b + a.\overline{b} + 0).(a + \overline{b})$
  $S2 = (\overline{a}.b + a.\overline{b}).(a + \overline{b})$
  $S2 = a.\overline{a}.b + \overline{a}.b.\overline{b} + a.a.\overline{b} + a.\overline{b}.\overline{b}$
  $S2 = 0 + 0 + a.\overline{b} + a.\overline{b}$
  $S2 = a.\overline{b}$

- $S3 = (a + \overline{b}).(a + b) + c.(\overline{a} + b)$
  $S3 = a.a + a.b + \overline{b}.a + \overline{b}.b + c.\overline{a} + c.b$
  $S3 = \mathbf{a} + \mathbf{a}.b + \overline{b}.\mathbf{a} + c.\overline{a} + c.b$ → *Theorem 1*
  $S3 = \mathbf{a} + c.\overline{\mathbf{a}} + c.b$ → *Theorem 2*
  $S3 = a + \mathbf{c} + \mathbf{c}.b$ → *Theorem 1*
  $S3 = a + c$

- $S4 = \overline{\overline{a}.\overline{c}.\overline{d} + \overline{b}.\overline{c}.\overline{d}}$
  $S4 = \overline{\overline{a}.\overline{c}.\overline{d}} \,.\, \overline{\overline{b}.\overline{c}.\overline{d}}$
  $S4 = (a + c + d).(b + c + d)$
  $S4 = a.b + a.c + a.d + c.b + c.c + c.d + d.b + d.c + d.d$
  $S4 = a.b + a.\mathbf{c} + a.d + \mathbf{c}.b + \mathbf{c} + \mathbf{c}.d + d.b + d.\mathbf{c} + d$ → *Theorem 1*
  $S4 = a.b + a.\mathbf{d} + c + \mathbf{d}.b + \mathbf{d}$ → *Theorem 1*
  $S4 = a.b + c + d$

- $S5 = (a + \overline{b}.\overline{c}).\overline{a.b}$
  $S5 = (a + \overline{b}.\overline{c}).(\overline{a} + \overline{b})$
  $S5 = a.\overline{a} + a.\overline{b} + \overline{b}.\overline{c}.\overline{a} + \overline{b}.\overline{c}.\overline{b}$
  $S5 = 0 + a.\overline{b} + \overline{b}.\overline{c}.\overline{a} + \overline{b}.\overline{c}$
  $S5 = a.\overline{b} + \mathbf{\overline{b}.\overline{c}}.\overline{a} + \mathbf{\overline{b}.\overline{c}}$ → *Theorem 1*
  $S5 = a.\overline{b} + \overline{b}.\overline{c}$

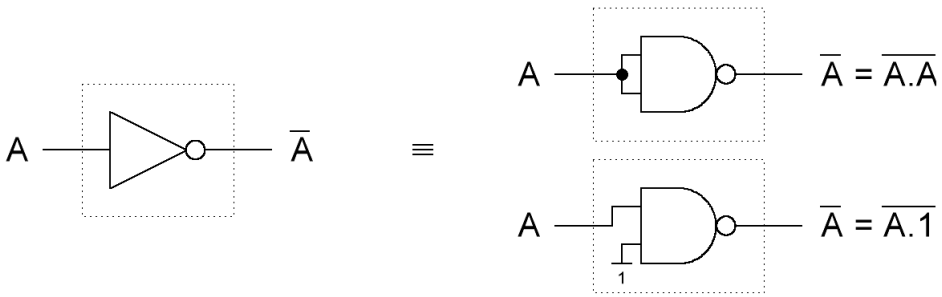
- $S6 = a.b.c + a.\overline{b}.\overline{c} + \overline{a}.b.\overline{c} + \overline{a}.b.c$
  $S6 = b.c.(a + \overline{a}) + a.\overline{b}.\overline{c} + \overline{a}.b.\overline{c}$
  $S6 = b.c.1 + a.\overline{b}.\overline{c} + \overline{a}.b.\overline{c}$
  $S6 = b.c + a.\overline{b}.\overline{c} + \overline{a}.b.\overline{c}$
  $S6 = b.(\mathbf{c} + \overline{a}.\mathbf{\overline{c}}) + a.\overline{b}.\overline{c}$ → *Theorem 2*
  $S6 = b.\mathbf{(c + \overline{a})} + a.\overline{b}.\overline{c}$ → *De Morgan's Theorem*
  $S6 = b.\overline{\overline{c}.a} + \overline{b}.\overline{c}.a$
  $S6 = b \oplus (\overline{c}.a)$

- $S7 = \mathbf{a} + b.c + \overline{\mathbf{a}}.(\overline{b} + \overline{c}).(a.d + c)$ → *Theorem 2*
  $S7 = a + b.c + \mathbf{(\overline{b} + \overline{c})}.(a.d + c)$ → *De Morgan's Theorem*
  $S7 = a + \mathbf{b.c} + \overline{\mathbf{b.c}}.(a.d + c)$ → *Theorem 2*
  $S7 = \mathbf{a} + b.c + \mathbf{a}.d + c$ → *Theorem 1*
  $S7 = a + b.\mathbf{c} + \mathbf{c}$ → *Theorem 1*
  $S7 = a + c$

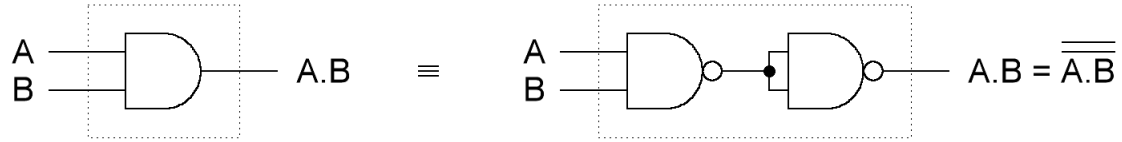
## Exercise 2

Design the NOT, AND, OR and XOR gates by using only NAND gates.

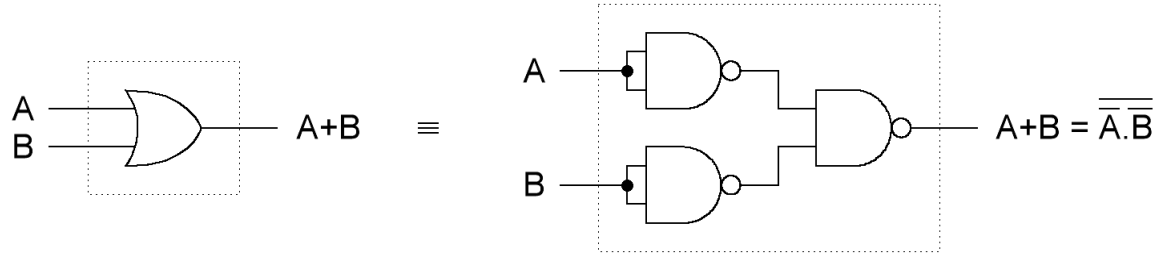
- NOT (two possibilities):



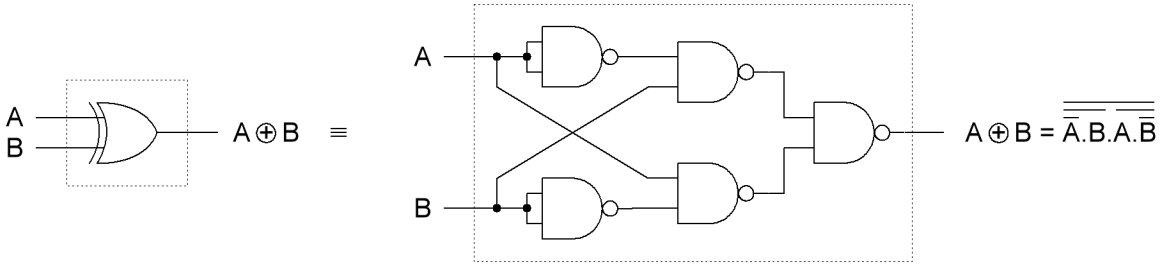
- AND



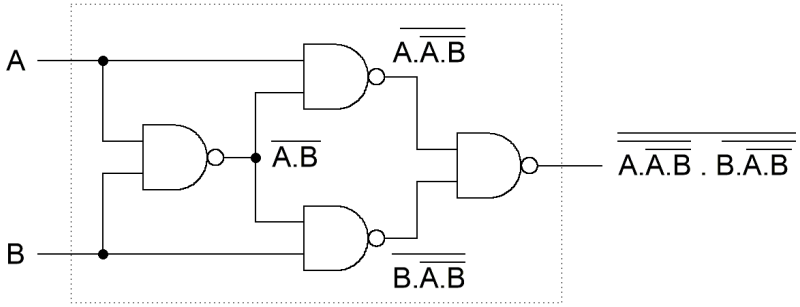

- OR




- XOR (with **five** NAND gates)




- XOR (with **four** NAND gates)
  - $\overline{A}.B = \overline{A}.B + B.\overline{B} = B.(\overline{A} + \overline{B}) = B.\overline{A.B}$
  - $A.\overline{B} = A.\overline{B} + A.\overline{A} = A.(\overline{A} + \overline{B}) = A.\overline{A.B}$



## Exercise 3

1. Write down the following expressions by using minterm canonical forms.

- $S1 = a.c.d + b.c.\overline{d} + \overline{b}.\overline{c}.d$
  $S1 = a.c.d.(b + \overline{b}) + b.c.\overline{d}.(a + \overline{a}) + \overline{b}.\overline{c}.d.(a + \overline{a})$
  $S1 = a.b.c.d + a.\overline{b}.c.d + a.b.c.\overline{d} + \overline{a}.b.c.\overline{d} + a.\overline{b}.\overline{c}.d + \overline{a}.\overline{b}.\overline{c}.d$

- $S2 = (\overline{a} + \overline{c}).(a + \overline{d} + c).b.\overline{c}$
  $S2 = (\overline{a}.a + \overline{a}.\overline{d} + \overline{a}.c + \overline{c}.a + \overline{c}.\overline{d} + \overline{c}.c ).b.\overline{c}$
  $S2 = (0 + \overline{a}.\overline{d} + \overline{a}.c + \overline{c}.a + \overline{c}.\overline{d} + 0).b.\overline{c}$
  $S2 = (\overline{a}.\overline{d} + \overline{a}.c + \overline{c}.a + \overline{c}.\overline{d}).b.\overline{c}$
  $S2 = \overline{a}.\overline{d}.b.\overline{c} + \overline{a}.c.b.\overline{c} + \overline{c}.a.b.\overline{c} + \overline{c}.\overline{d}.b.\overline{c}$
  $S2 = \overline{a}.b.\overline{c}.\overline{d} + 0 + a.b.\overline{c} + b.\overline{c}.\overline{d}$
  $S2 = \overline{a}.b.\overline{c}.\overline{d} + a.b.\overline{c} + b.\overline{c}.\overline{d}$
  $S2 = \overline{a}.b.\overline{c}.\overline{d} + a.b.\overline{c}.(d + \overline{d}) + b.\overline{c}.\overline{d}.(a + \overline{a})$
  $S2 = \mathbf{\overline{a}.b.\overline{c}.\overline{d}} + a.b.\overline{c}.d + \mathbf{a.b.\overline{c}.\overline{d}} + \mathbf{a.b.\overline{c}.\overline{d}} + \mathbf{\overline{a}.b.\overline{c}.\overline{d}}$ → *Remove duplication*
  $S2 = \overline{a}.b.\overline{c}.\overline{d} + a.b.\overline{c}.d + a.b.\overline{c}.\overline{d}$

- $S3 = b.c.(a + \overline{d}) + \overline{b}.d.(a + \overline{c})$
  $S3 = b.c.a + b.c.\overline{d} + \overline{b}.d.a + \overline{b}.d.\overline{c}$
  $S3 = b.c.a.(d + \overline{d}) + b.c.\overline{d}.(a + \overline{a}) + \overline{b}.d.a.(c + \overline{c}) + \overline{b}.d.\overline{c}.(a + \overline{a})$
  $S3 = a.b.c.d + \mathbf{a.b.c.\overline{d}} + \mathbf{a.b.c.\overline{d}} + \overline{a}.b.c.\overline{d} + a.\overline{b}.c.d + \mathbf{a.\overline{b}.\overline{c}.d} + \mathbf{a.\overline{b}.\overline{c}.d} + \overline{a}.\overline{b}.\overline{c}.d$ → *Remove dup.*
  $S3 = a.b.c.d + a.b.c.\overline{d} + \overline{a}.b.c.\overline{d} + a.\overline{b}.c.d + a.\overline{b}.\overline{c}.d + \overline{a}.\overline{b}.\overline{c}.d$

2. Write down the following expressions by using maxterm canonical forms.

- $S1 = (a + c).(\overline{a} + b + c)$
  $S1 = (a + b.\overline{b} + c).(\overline{a} + b + c)$ → *Distributivity*
  $S1 = (a + b + c).(a + \overline{b} + c).(\overline{a} + b + c)$

- $S2 = a.b + a.\overline{c} + \overline{a}.\overline{b}.c$ → *Distributivity*
  $S2 = (\mathbf{a}.b + \mathbf{a}.\overline{c} + \mathbf{\overline{a}}).(a.\mathbf{b} + a.\overline{c} + \mathbf{\overline{b}}).(a.b + a.\overline{\mathbf{c}} + \mathbf{c})$ → *Theorem 2*
  $S2 = (b + \overline{c} + \overline{a}).(\mathbf{a} + \mathbf{a}.\overline{c} + \overline{b}).(\mathbf{a}.b + \mathbf{a} + c)$ → *Theorem 1*
  $S2 = (\overline{a} + b + \overline{c}).(a + \overline{b}).(a + c)$
  $S2 = (\overline{a} + b + \overline{c}).(a + \overline{b} + c.\overline{c}).(a + b.\overline{b} + c)$ → *Distributivity*
  $S2 = (\overline{a} + b + \overline{c}).\mathbf{(a + \overline{b} + c)}.(a + \overline{b} + \overline{c}).(a + b + c).\mathbf{(a + \overline{b} + c)}$ → *Remove duplication*
  $S2 = (\overline{a} + b + \overline{c}).(a + \overline{b} + c).(a + \overline{b} + \overline{c}).(a + b + c)$

## Exercise 4

Prove that the following identities are true.

- $\overline{\mathbf{a.c} + \mathbf{b.\bar{c}}} = \bar{a}.c + \bar{b}.\bar{c}$ → *De Morgan's Theorem*

  $\overline{\mathbf{a.c}} \;.\; \overline{\mathbf{b.\bar{c}}} = \bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$ → *De Morgan's Theorem*

  $(\bar{a} + \bar{c}).(\bar{b} + c) = \bar{a}.c + \bar{b}.\bar{c}$

  $\bar{a}.\bar{b} + \bar{a}.c + \bar{c}.\bar{b} + \bar{c}.c = \bar{a}.c + \bar{b}.\bar{c}$

  $\bar{a}.\bar{b} + \bar{a}.c + \bar{c}.\bar{b} + 0 = \bar{a}.c + \bar{b}.\bar{c}$

  $\bar{a}.\bar{b} + \bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$

  $\bar{a}.\bar{b}.\mathbf{(c + \bar{c})} + \bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$

  $\bar{\mathbf{a}}.\bar{b}.\mathbf{c} + \bar{a}.\bar{b}.\bar{c} + \mathbf{a.c} + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$ → *Theorem 1*

  $\bar{a}.\mathbf{\bar{b}.c} + \bar{a}.c + \mathbf{\bar{b}.\bar{c}} = \bar{a}.c + \bar{b}.\bar{c}$ → *Theorem 1*

  $\bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$

- $\mathbf{(a + b).(\bar{a} + c)}.(b + c) = \mathbf{(a + b).(\bar{a} + c)}$
- Let us prove that we can ignore (b + c):
  - The maxterm canonical form of $(a + b).(\bar{a} + c)$ is $\mathbf{(a + b + c)}.(a + b + \bar{c}).\mathbf{(\bar{a} + b + c)}.(\bar{a} + \bar{b} + c)$.
  - The maxterm canonical form of (b + c) is $\mathbf{(a + b + c).(\bar{a} + b + c)}$ and these two sums are already in the canonical form of $(a + b).(\bar{a} + c)$.
  - Therefore, we can ignore them in the canonical form of $(a + b).(\bar{a} + c).(b + c)$.
  - The canonical form of $(a + b).(\bar{a} + c).(b + c)$ is equal to the canonical form of $(a + b).(\bar{a} + c)$.
  - When the canonical forms of two expressions are equal, it means that the two expressions are equal.

## Exercise 5

Let us consider the following binary variables: A, B, C. Write down an expression that is 1 when the number of variables being 1 is odd (simplify with EXCLUSIVE OR).

| A | B | C | Output | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 0 | 0 | 1 | 1 | → $\bar{A}.\bar{B}.C$ |
| 0 | 1 | 0 | 1 | → $\bar{A}.B.\bar{C}$ |
| 0 | 1 | 1 | 0 | |
| 1 | 0 | 0 | 1 | → $A.\bar{B}.\bar{C}$ |
| 1 | 0 | 1 | 0 | |
| 1 | 1 | 0 | 0 | |
| 1 | 1 | 1 | 1 | → $A.B.C$ |

$\text{Output} = \bar{A}.\bar{B}.C + \bar{A}.B.\bar{C} + A.\bar{B}.\bar{C} + A.B.C$

$\text{Output} = \bar{A}.(\bar{B}.C + B.\bar{C}) + A.(\bar{B}.\bar{C} + B.C)$

$\text{Output} = \bar{A}.(B \oplus C) + A.(\overline{B \oplus C})$

$\mathbf{Output = A \oplus B \oplus C}$